Momentum, angular momentum and their quasi-local null surface extensions

1983 J. Phys. A: Math. Gen. 16 1155

(http://iopscience.iop.org/0305-4470/16/6/010)

J. Phys. A: Math. Gen. **16** (1983) 1155–1168. Printed in Great Britain

Momentum, angular momentum and their quasi-local null surface extensions

M Ludvigsen† and J A G Vickers

Department of Mathematics, University of York, Heslington, York YO1 5DD, England

Received 7 October 1982

Abstract. Let M be an asymptotically flat space–time which satisfies the dominant energy condition and let $\mathcal{N}$ be a null hypersurface of M which meets $\mathscr{I}^+$ in a space-like cross section S given by $r=\infty$, where r is an affine parameter along the generators of $\mathcal{N}$. It is shown that there exists a quasi-local momentum $P_a(r)$, defined by means of an integral over the $r=$ constant cross sections of $\mathcal{N}$, which tends to the Bondi momentum as $r\to\infty$, and which satisfies the radial 'mass-gain condition' $P_a(r_2)k^a \geq P_a(r_1)k^a$ when $r_2 \geq r_1$, where k^a is a constant future pointing null vector. This condition is then used to show that, in certain circumstances, the Bondi momentum is always future pointing or, in other words, that the Bondi mass is always positive. A quasi-local angular momentum is also defined, which tends to Bramson's angular momentum as $r\to\infty$.

1. Introduction

Some twenty years ago Bondi *et al* (1962) introduced a definition of momentum for a null asymptotically flat space–time which has since become known as Bondi momentum. Unlike the ADM four-momentum which is time independent, the Bondi momentum is a four-vector function which is defined on the space of all space-like cross sections (cuts) of $\mathscr{I}^+$ and which lies in the Minkowski space of BMS translations. Given any cut S of $\mathscr{I}^+$, the Bondi four-momentum $P_a(S)$ may be interpreted as the total energy–momentum of the space–time at the retarded time defined by S. One of the most important properties of the Bondi mass is that it satisfies the 'mass-loss formula' on $\mathscr{I}^+$. In terms of the Bondi four-momentum this may be stated as follows: given two arbitrary cuts S_1 and S_2 of $\mathscr{I}^+$ with S_2 entirely to the future of S_1 and some future pointing vector k^a, then $P_a(S_2)k^a \leq P_a(S_1)k^a$ provided the space–time satisfies the weak energy condition in a neighbourhood of $\mathscr{I}^+$. The mass-loss condition is important because it shows that gravitational radiation emitted by an isolated system carries positive mass, but it says nothing about the total mass of the system. It has been a long standing conjecture in general relativity that for an asymptotically flat space–time which satisfies an appropriate energy condition the Bondi mass is positive (i.e. $P_a(S)$ is future pointing). This is usually referred to as the positive mass conjecture at null infinity.

† Present address: Department of Mathematics, University of Canterbury, Christchurch, New Zealand.

In a number of recent papers this conjecture has been confirmed. One method has involved the use of topological arguments similar to those used by Schoen and Yau (1981) in their proof of the positivity of the ADM mass, while another method has involved the use of spinor techniques (see e.g. Ludvigsen and Vickers 1982 or Horowitz and Perry 1982). This second approach has been largely based upon Witten's proof (1981) of the positivity of the ADM mass, but replaces the space-like hypersurfaces by one that is space-like and asymptotically null. The difficulty of this method lies in showing the existence of an asymptotically constant spinor field which satisfies a certain differential equation. In this paper we adopt a slightly different approach. Rather than considering space-like and asymptotically null hypersurfaces, we restrict our attention to null hypersurfaces. On such hypersurfaces it becomes possible to replace the Witten equation by a much simpler first-order propagation equation, which is easily seen to possess a solution with the required asymptotic behaviour. While problems concerning the development of caustics prevent this method from giving a general proof of the positivity of the Bondi mass, it does provide a simple and direct proof of positivity in a number of physically interesting situations. Furthermore, this propagation equation, unlike the Witten equation, has the additional property that it may be used to give a simple expression for the total angular momentum.

In § 3 we will show how these methods may be used to give a quasi-local definition of momentum in which a total energy–momentum $P_a(S)$ is assigned to a two-surface S bounding a compact portion H of a space-like hypersurface. This definition has several desirable properties. Firstly, in the limit that S becomes a cut of $\mathscr{I}^+$, $P_a(S)$ becomes the Bondi momentum; secondly, if S_2 is obtained by moving S_1 out along the null geodesics orthogonal to S_1, then $P_a(S_2)k^a \geqslant P_a(S_1)k^a$; and thirdly, the definition reduces to the correct one for linearised gravity.

We also show how the same methods may be used to provide a quasi-local definition of angular momentum $S_{ab}(S)$. This definition has the properties that in the limit that S becomes a cut of $\mathscr{I}^+$, $S_{ab}(S)$ becomes Bramson's definition of angular momentum, and that it reduces to the correct result for linearised gravity.

2. Bondi momentum

Let M be an asymptotically flat space–time with future null infinity $\mathscr{I}^+$ and let $\mathcal{N}$ be an outgoing null hypersurface in M which meets $\mathscr{I}^+$ in some space-like cross section S_∞. The Bondi momentum $P_a(S_\infty)$ may be considered as an abstract four-vector in the Minkowski BMS vector space of translations M_0 and represents the total momentum of the space at the retarded time given by $\mathcal{N}$. If we write $M_0 = \mathscr{S} \otimes \bar{\mathscr{S}}$, where $\mathscr{S}$ is the two-spinor space which Bramson (1976) calls the asymptotic spin space of M, then on using the abstract index notation of Penrose (1968), we may write $P_a = P_{\mathbf{A}\mathbf{A}'}$ where the indices $\mathbf{A}$ and $\mathbf{A}'$ refer to $\mathscr{S}$ and $\bar{\mathscr{S}}$ respectively. In this section we shall first give an abstract coordinate-free definition of $P_{\mathbf{A}\mathbf{A}'}$ and then show how this may be naturally generalised to give a quasi-local definition of the total energy-momentum $P_a(S)$ associated with a portion of space–time bounded by some finite space-like cross sections S of $\mathcal{N}$.

We begin by describing the asymptotic spin space $\mathscr{S}$ and its relationship to asymptotic spinor fields on $\mathcal{N}$. Let r be an affine parameter along the null generators of $\mathcal{N}$ which tends to $+\infty$ as S_∞ is approached. As all of the following analysis is invariant under a change of affine parameter, it is not necessary to impose any further restriction upon its choice. We shall not, for instance, demand that r be a Bondi radial coordinate. Given r, we now introduce two null vector fields l_a and n_a on $\mathcal{N}$ which are normalised such that $l_a n^a = 1$ with l_a pointing along the generators and satisfying $l^a = \partial x^a/\partial r$ and with n^a pointing out of and orthogonal to the $r = \text{constant}$ cuts of $\mathcal{N}$. Under these conditions l_a and n_a are defined uniquely and furthermore l_a satisfies $\text{Þ} l_a = 0$ (where $\text{Þ} := l^a\nabla_a$). Since l_a and n_a are null and satisfy $l_a n^a = 1$ we may write them in the form $l_a = o_A o_{A'}$ and $n_a = \iota_A \iota_{A'}$ where the spinors o_A and ι_A satisfy $o_A \iota^A = 1$ and $\text{Þ} o_A = 0$.

A spinor field λ_A on $\mathcal{N}$ is asymptotically covariantly constant if $o^A\lambda_A$ and $\iota^A\lambda_A$ remain finite as $r \to \infty$ and in addition

$$\lim_{r\to\infty} r\iota^A o^{A'} o^B \nabla_{AA'}\lambda_B = \lim_{r\to\infty} r o^A \iota^{A'} o^B \nabla_{AA'}\lambda_B = 0 \tag{1}$$

and

$$\lim_{r\to\infty} r\iota^A o^{A'} \iota^B \nabla_{AA'}\lambda_B = \lim_{r\to\infty} r o^A \iota^{A'} \iota^B \nabla_{AA'}\lambda_B = 0. \tag{2}$$

However, as Bramson has pointed out, equations (1) and (2) are inconsistent for radiating space-times, so that in the presence of radiation, asymptotically constant spinor fields on $\mathcal{N}$ do not exist. Nevertheless it is possible to give a weaker condition of asymptotic constancy by demanding that equation (1) alone is satisfied, and this is consistent even in the presence of radiation. This weaker condition is reasonable for the following reasons. Firstly, in the absence of radiation, it reduces to the stronger condition (i.e. when there is no radiation equation (1) implies equation (2)), and secondly because it implies, in general, that the field possesses asymptotically two complex degrees of freedom. Thus if $X^A_{\mathbf{A}}$ ($\mathbf{A} = 0, 1$) is an asymptotically constant spinor basis satisfying

$$\lim_{r\to\infty} X_{0A} X_1^A = 1 \tag{3}$$

and λ_A is an arbitrary asymptotically constant spinor field, then the asymptotic spinor component $\overset{0}{\lambda}_{\mathbf{A}}$ of λ_A with respect to $X^A_{\mathbf{A}}$, which is given by

$$\overset{0}{\lambda}_{\mathbf{A}} = \lim_{r\to\infty} (X^A_{\mathbf{A}}\lambda_A) \tag{4}$$

is such that $\overset{0}{\lambda}_{\mathbf{A}}$ is constant. In what follows it will be useful to consider $\mathbf{A}$ as an abstract index referring to an abstract spinor space $\mathcal{S}$, the asymptotic spin space, and equation (4) as giving a mapping from asymptotically constant spinor fields into $\mathcal{S}$. Thus, for each asymptotically constant spinor field there exists a single spinor $\overset{0}{\lambda}_{\mathbf{A}}$, its asymptotic component, in $\mathcal{S}$. From (3) and (4) it is clear that

$$\lim_{r\to\infty} (\lambda_A \mu^A) = \overset{0}{\lambda}_{\mathbf{A}} \overset{0}{\mu}{}^{\mathbf{A}}.$$

We are now in a position to give an abstract definition of the Bondi momentum $P_{\mathbf{AA'}}(S_\infty)$ with respect to S_∞. Let λ_A be an arbitrary asymptotically constant spinor field and let

$$\varphi_{AB} = \tfrac{1}{2}\lambda_{(A}\nabla_{B)}{}^{C'}\lambda_{C'} - \tfrac{1}{2}\lambda_{C'}\nabla_{(A}{}^{C'}\lambda_{B)}. \tag{5}$$

Then

$$P_{AA'}(S_\infty)\overset{0}{\lambda}{}^{A}\overset{0}{\lambda}{}^{A'} = \lim_{r\to\infty} \oint_{S_r} \varphi_{AB} o^{A} \iota^{B}\, dS + \text{CC} \tag{6}$$

where CC stands for complex conjugate and dS is the surface element of S_r, the r = constant cut of $\mathcal{N}$.

In vector form (6) gives

$$P_a(S_\infty)k^a = \lim_{r\to\infty} \oint_{S_r} F_{ab}\, d\Sigma^{ab} \tag{7}$$

where

$$F_{ab} = \varphi_{AB}\varepsilon_{A'B'} + \text{CC} \tag{8}$$

$k^a = \overset{0}{\lambda}{}^{A}\overset{0}{\lambda}{}^{A'}$ and $d\Sigma^{ab} = l^{[a}n^{b]}\, dS$ is the surface form of S_r, the r = constant cut of $\mathcal{N}$.

From equation (7) it is clear that P_a is independent of the choice of affine parameter r. That (6) (or (7)) does indeed yield the standard expression for P_a can be seen by referring everything to a Bondi-type coordinate system. In this case (see appendix) (6) gives

$$P_{AA'}(S_\infty) = -\tfrac{1}{2} \oint (\overset{0}{\psi}_2 + \sigma^0 \dot{\bar{\sigma}}^0) o_A o_{A'}\, dS_0. \tag{9}$$

3. Quasi-local momentum

Up to now, apart from satisfying (1), we have placed no restriction upon the spinor field λ_A; only its asymptotic form is determined by specifying $\overset{0}{\lambda}_A$. We shall now restrict the spinor field further by requiring that it satisfies a first-order propagation equation, which reduces to parallel propagation for a flat space–time, and which determines the whole of the field λ_A once its asymptotic component $\overset{0}{\lambda}_A$ is given. Once this equation is imposed the integral

$$I(r) = \oint_{S_r} \varphi_{AB} o^{A} \iota^{B}\, dS + \text{CC} \tag{10}$$

is linear in $\overset{0}{\lambda}{}^{A}\overset{0}{\lambda}{}^{A'}$ and thus determines a quasi-local momentum $P_{AA'}(S_r) \in \mathcal{S} \otimes \mathcal{S}$ according to

$$P_{AA'}(S_r)\overset{0}{\lambda}{}^{A}\overset{0}{\lambda}{}^{A'} = I(r). \tag{11}$$

$P_{AA'}$ will however depend crucially upon the choice of propagation equation. The particular propagation law that we impose upon λ_A appears to be the natural choice in the sense that it satisfies the following conditions: (i) it reduces to parallel propagation when M is flat; (ii) it implies that the corresponding momentum satisfies the mass-gain formula $P_a(S_{r_2})k^a \geqslant P_a(S_{r_1})k^a$ for $r_2 \geqslant r_1$ provided the dominant energy condition holds on $\mathcal{N}$; (iii) in the case of linearised theory $P_a(S)$ reduces to the 'correct' expression for the total momentum contained within the surface S.

A remarkable property of F_{ab} given by (8) is that it satisfies the relation

$$\nabla^b F_{ab} = \tfrac{1}{2}(\nabla^B{}_{A'}\lambda_{C'}\nabla_{(B}{}^{C'}\lambda_{A)} - \nabla^B{}_{A'}\lambda_{(B}\nabla_{A)}{}^{C'}\lambda_{C'} + \mathrm{CC}) - \tfrac{1}{2}G_{ab}k^b \tag{12}$$

where $k_a = \lambda_A \lambda_{A'}$ and G_{ab} is the Einstein tensor. This result can be proved by using the spin coefficient form of the Ricci identity

$$\nabla_{[c}\nabla_{d]}T_b = \tfrac{1}{2}R^a{}_{bcd}T_a \tag{13}$$

(this equation, incidentally, fixes the sign of our Riemann tensor). Thus, on using a null hypersurface form of Gauss's theorem (see appendix) based on $\mathcal{N}$ with affine parameter r, we have

$$\begin{aligned} I(r_2) - I(r_1) &= \int_{r=r_1}^{r_2} \oint_{S_r} l^a \nabla^b F_{ab}\, \mathrm{d}S\, \mathrm{d}r \\ &= \tfrac{1}{2}\int_{r=r_1}^{r_2} \oint_{S_r} [(\nabla^B{}_{A'}\lambda_{C'}\nabla_{(B}{}^{C'}\lambda_{A)} - \nabla^B{}_{A'}\lambda_{(B}\nabla_{A)}{}^{C'}\lambda_{C'} + \mathrm{CC})o^A o^{A'} \\ &\quad - G_{ab}l^a k^b]\, \mathrm{d}S\, \mathrm{d}r. \end{aligned} \tag{14}$$

If we now restrict λ_A to satisfy the propagation equation

$$o^A o^{A'} \nabla_{BA'}\lambda_A = 0, \tag{15}$$

then, after some manipulation, it can be shown that (14) becomes

$$I(r_2) - I(r_1) = \int_{r=r_1}^{r_2} \oint_{S_r} (X\bar{X} - \tfrac{1}{2}G_{ab}l^a k^b)\, \mathrm{d}S\, \mathrm{d}r \tag{16}$$

where

$$X := o^A \iota^{A'} o^B \nabla_{BA'}\lambda_A. \tag{17}$$

Thus provided that the dominant energy condition holds, $G_{ab}l^a k^b \leqslant 0$, the integrand in (16) is non-negative and thus

$$I(r_2) \geqslant I(r_1) \qquad \text{for } r_2 \geqslant r_1. \tag{18}$$

From this it can be seen that if (15) does indeed determine λ_A over the whole of $\mathcal{N}$ once its asymptotic component is specified, then (10), (11) and (15) will define a quasi-local momentum satisfying condition (ii). We now proceed to show that this is actually the case.

In terms of the GHP spin coefficient notation (Geroch *et al* 1973) equation (15) may be written

$$Þ\lambda_0 = \partial\lambda_0/\partial r = 0 \tag{19}$$

$$\bar{ð}\lambda_0 + \rho\lambda_1 = 0 \tag{20}$$

where λ_0 and λ_1 are the components of λ_A with respect to o_A and ι_A, ρ is the divergence of $\mathcal{N}$, and $\bar{ð}$ is the 'angular' GHP edth operator. From these two equations it is clear that once λ_A is specified on some $r =$ constant cut, then it is defined over the whole of $\mathcal{N}$: equation (19) determines λ_0 on $\mathcal{N}$ and thus in turn determines λ_1 by (20). Furthermore, it is easily seen that these equations are compatible with the asymptotic conditions given by (1). Thus a spinor field satisfying (15) is uniquely determined over the whole of $\mathcal{N}$ once its asymptotic component $\overset{0}{\lambda}_A$ is specified. Since equation (15) is obviously satisfied by a parallelly propagated spinor field when M is flat we have, by uniqueness, that the field specified by a given $\overset{0}{\lambda}_A$ is indeed parallelly propagated over $\mathcal{N}$ when M is flat. Thus the propagation law given by (15) satisfies condition (i). We conclude this section by considering condition (iii).

Let M be a linear space-time with metric $g_{ab} = \eta_{ab} + h_{ab}$, with h_{ab} small and $O(\varepsilon)$, say, which satisfies the linearised Einstein equation

$$G_{ab} = -\kappa T_{ab} \tag{21}$$

where κ is $O(\varepsilon)$. We shall denote all small quantities by $O(\varepsilon)$ and neglect products of such quantities. Let $\mathcal{N}$ be a null hypersurface with respect to η_{ab} and $\mathcal{N}'$ a corresponding hypersurface which is null with respect to g_{ab}. Then, with an abuse of notation, we have

$$\mathcal{N} - \mathcal{N}' = O(\varepsilon). \tag{22}$$

Let λ_A, λ'_A be spinor fields on $\mathcal{N}$ and $\mathcal{N}'$ respectively, with both having the same asymptotic limit but with λ_A propagated according to (15) and λ'_A parallelly propagated. Thus

$$\lambda_A - \lambda'_A = O(\varepsilon). \tag{23}$$

Since X defined by (17) vanishes in the case of a flat space-time we have

$$X = O(\varepsilon). \tag{24}$$

Substituting all this information into (16) we obtain

$$P_a(r_2) - P_a(r_1) = \tfrac{1}{2}\kappa \int\oint T_{ab} l^a k^b \, dS \, dr \tag{25}$$

which gives the correct value in flat space-time for the total energy momentum contained between the cuts $r = r_1$ and $r = r_2$.

4. Applications

As was mentioned in the introduction, the almost inevitable occurrence of caustics on the null hypersurface $\mathcal{N}$ imposes a general restriction on the large-scale applicability of our methods and prevents us from obtaining a proof of the positivity of the Bondi mass for a general space-time. Nevertheless, in a number of physically interesting special situations, such as a space-time containing a single black hole, our methods can be used to show that the Bondi mass is positive. We shall in this section consider these situations.

On expressing (10) in spin coefficient notation based on the GHP formalism, using equation (20) and integrating by parts, the quantity $I(r)$ can be written as

$$I(r) = -\oint_{S_r} (\rho' \lambda_0 \bar{\lambda}_0 + \rho \lambda_1 \bar{\lambda}_1) \, dS \tag{26}$$

where ρ is the divergence of $\mathcal{N}$ and ρ' is the divergence of the ingoing null hypersurface $\mathcal{N}'$ which meets S_r orthogonally. With our choice of tetrad, l^a and n^a are null tangents to $\mathcal{N}$ and $\mathcal{N}'$ respectively and thus ρ and ρ' are both real and given by

$$\rho = \iota^A \bar{o}^{A'} o^B \nabla_{AA'} o_B \tag{27}$$

$$\rho' = -o^A \bar{\iota}^{A'} \iota^B \nabla_{AA'} \iota_B. \tag{28}$$

We consider first the rather restricted situation where $\mathcal{N}$ has no caustic surfaces and converges to a single point, which we represent by O. If we choose r such that it vanishes at O then, since $\mathcal{N}$ is the null cone emanating from O, we have for small r

$$\rho \sim -(1/r) \qquad \rho' \sim 1/r \qquad \text{and} \qquad \mathrm{d}S \sim r^2\,\mathrm{d}S_0$$

so that $I(r)$ is order r and thus

$$P_a k^a = \lim_{r_2\to\infty}\lim_{r_1\to 0}\int_{r_1}^{r_2}\oint_{S_r}(X\bar{X} - \tfrac{1}{2}G_{ab}k^a l^b)\,\mathrm{d}S\,\mathrm{d}r \geqslant 0. \tag{29}$$

Since the above is true for all null vectors $k^a = \overset{0}{\lambda}{}^{A}\overset{0}{\bar{\lambda}}{}^{A'}$ (which are necessarily future pointing) this implies that P_a is future pointing or, in other words, the Bondi mass is positive. The condition that $\mathcal{N}$ converges to a point is, of course, rather severe and therefore the result is in itself of limited value.

We consider next the considerably more general situation where $\mathcal{N}$ is allowed to develop caustics but where there exists an ingoing null hypersurface $\mathcal{N}'$ which does converge to a point, O say, and which meets $\mathcal{N}$ in a non-singular space-like cut S_1 (see figure 1). Let r be an affine parameter on $\mathcal{N}$ such that $r = r_1$ on S_1, and let r' be an affine parameter on $\mathcal{N}'$ such that $r' = r'_1$ on S_1 and $r' = 0$ at O. From (16) we have

$$P_a(S_\infty)k^a \geqslant I(r_1) \tag{30}$$

and, by adapting our previous arguments to an ingoing null hypersurface $\mathcal{N}'$, one can show that

$$I'(r'_1) = \int_{r'=0}^{r'_1}(\rho'\lambda'_0\bar{\lambda}'_0 + \rho'_1\bar{\lambda}'_1)\,\mathrm{d}S \geqslant 0 \tag{31}$$

where λ'_A is a spinor field on $\mathcal{N}'$ satisfying the propagation law

$$\iota^A\bar{\iota}^{A'}\nabla_{BA'}\lambda_A = 0. \tag{32}$$

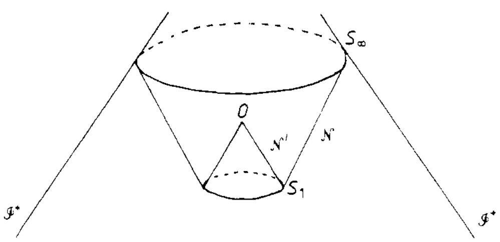

Figure 1.

In terms of spin coefficients (32) may be written

$$\text{þ}'\lambda'_1 = 0 \tag{33}$$

$$\eth\lambda'_1 + \rho'\lambda'_0 = 0. \tag{34}$$

Thus, in order to show that $P_a k^a \geqslant 0$, it remains to show that λ_A may be related to λ'_A on S_1 in such a way that $I(r_1) \geqslant I'(r'_1)$. We do this by demanding that

$$\lambda_0 = \lambda'_0 \qquad \text{and} \qquad \eth\lambda'_1 + \rho'\lambda_0 = 0 \qquad \text{on } S_1. \tag{35}$$

These conditions are compatible with (31) and thus uniquely determine λ'_A over the whole of $\mathcal{N}'$. On S_1 we have

$$I'_1 = \oint_{S_1} (\rho'\lambda'_0\bar{\lambda}'_0 + \rho\lambda'_1\bar{\lambda}'_1)\,\mathrm{d}S$$

$$= \oint_{S_1} [-(\eth\lambda'_1) - \bar{\lambda}_0(\bar{\eth}\bar{\lambda}'_1) - \rho'\lambda_0\bar{\lambda}_0 + \rho\lambda'_1\bar{\lambda}'_1]\,\mathrm{d}S$$

$$= \oint_{S_1} [\lambda'_1\eth\bar{\lambda}_0 + (\bar{\eth}\lambda_0)\bar{\lambda}'_1 + \rho\lambda'_1\bar{\lambda}'_1 - \rho'\lambda_0\bar{\lambda}_0]\,\mathrm{d}S$$

$$= \oint_{S_1} (-\rho\lambda'_1\bar{\lambda}_1 - \rho\lambda_1\bar{\lambda}'_1 + \rho\lambda'_1\bar{\lambda}'_1 - \rho'\lambda_0\bar{\lambda}_0)\,\mathrm{d}S$$

$$= \oint_{S_1} [-\rho(Y+\lambda_1)\bar{\lambda}_1 - \rho\lambda_1(\bar{Y}+\bar{\lambda}_1) + \rho(Y+\lambda_1)(\bar{Y}+\bar{\lambda}_1) - \rho'\lambda_0\bar{\lambda}_0]\,\mathrm{d}S$$

where $Y := \lambda'_1 - \lambda_1$

$$= \oint_{S_1} (-\rho'\lambda_0\bar{\lambda}_0 - \rho\lambda_1\bar{\lambda}_1 + \rho Y\bar{Y})\,\mathrm{d}S.$$

Thus

$$I'_1 = I_1 + \oint_{S_1} \rho Y\bar{Y}\,\mathrm{d}S \tag{36}$$

where we have integrated by parts and used (20), (34) and (35). From the well known spin coefficient equation

$$\partial\rho/\partial r = \text{Þ}\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00} \tag{37}$$

together with the dominant energy condition which implies that $\Phi_{00} \geqslant 0$, we see that $\partial\rho/\partial r \geqslant 0$. Thus, since $\rho \to 0$ as $r \to \infty$ on a null hypersurface which meets $\mathscr{I}^+$, we have $\rho \leqslant 0$ on S_1 and (36) gives

$$I_1 \geqslant I'_1. \tag{38}$$

Putting (30), (31) and (38) together we finally have

$$P_a(S_\infty)k^a \geqslant I_1 \geqslant I'_1 \geqslant 0 \tag{39}$$

so that $P_a(S_\infty)$ is thus future pointing.

Finally, we consider the case where M contains a single black hole and where $\mathcal{N}'$, instead of converging to a point, meets a trapped surface T (see figure 2) on which, by definition, $\rho > 0$ and $\rho' > 0$. In this case we have

$$P_a(S_\infty)k^a \geqslant I_1 \geqslant I'_1 \geqslant I'(T) \tag{40}$$

where

$$I(T) = \oint_T (\rho\lambda'_0\bar{\lambda}'_0 + \rho'\lambda'_1\bar{\lambda}'_1)\,\mathrm{d}S. \tag{41}$$

Since ρ and ρ' are both positive, $I'(T) > 0$ and thus by (40) P_a is again future pointing.

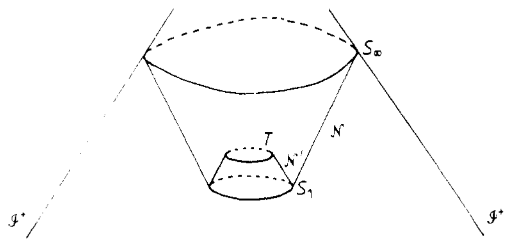

Figure 2.

5. Angular momentum

In this section we will show how the methods used in giving a quasi-local definition of momentum may also be used to give a quasi-local definition of angular momentum. We start by considering the expression

$$\omega_{\boldsymbol{AB}}(S_\infty)\overset{0}{\lambda}{}^{\boldsymbol{A}}\overset{0}{\lambda}{}^{\boldsymbol{B}} = \lim_{r\to\infty}\oint_{S_r} \lambda_{(A}\lambda_{B)}o^A\iota^B \, \mathrm{d}S \tag{42}$$

where λ_A is determined over the whole of $\mathscr{N}$ by specifying its asymptotic component $\overset{0}{\lambda}{}^{\boldsymbol{A}}$ and demanding that λ_A satisfies the propagation equation (15) $o^A o^{A'}\nabla_{BA}\lambda_A = 0$ just as in § 3.

In terms of a Bondi-type coordinate system (u, r, θ, ϕ) it may be shown that $\mathrm{d}S$, the surface element of the $r = \text{constant}$ cuts of $\mathscr{N}$, is given by

$$\mathrm{d}S = \tfrac{1}{2}r^2(1-\sigma^0\bar{\sigma}^0 r^{-2})\,\mathrm{d}S_0 + \mathrm{O}(r^{-1}). \tag{43}$$

In order to calculate the RHS of (42) it is therefore necessary to obtain asymptotic expansions for λ_0 and λ_1 as far as terms of order r^{-2}. In terms of spin coefficients based upon the spinor dyad (o_A, ι_A) introduced in § 2, equation (15) may be written

$$D\lambda_0 = 0 \tag{44}$$

$$\bar{\delta}\lambda_0 - \alpha\lambda_0 = -\rho\lambda_1. \tag{45}$$

Since in the present coordinate system D is just $\partial/\partial r$, (44) immediately gives $\lambda_0(r, \theta, \phi) = \overset{0}{\lambda}_0(\theta, \phi)$. In order to obtain an asymptotic expansion for λ_1 we use the fact that

$$\begin{aligned}\bar{\delta} &= \xi^i\partial/\partial x^i\\ &= \left(\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi}\right)r^{-1} - \sigma^0\left(\frac{\partial}{\partial\theta} + \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi}\right)r^{-2} + \sigma^0\bar{\sigma}^0\left(\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\phi}\right)r^{-3} + \mathrm{O}(r^{-4})\end{aligned}$$

$$\alpha = \alpha^0 r^{-1} + (\bar{\sigma}^0\bar{\alpha}^0 - \bar{\eth}_0\bar{\sigma}^0)r^{-2} + (\sigma^0\bar{\sigma}^0\alpha^0 + \bar{\sigma}^0\bar{\eth}_0\sigma^0 - \tfrac{1}{2}\bar{\psi}_1^0)r^{-3} + \mathrm{O}(r^{-4})$$

$$\rho = -r^{-1} - \sigma^0\bar{\sigma}r^{-3} + \mathrm{O}(r^{-5})$$

where $\alpha^0 = \frac{1}{2}\cot\theta$ and $\eth_0$ is the Newman–Penrose 'edth' operator on a metric sphere†.

† *Note.* These expansions are easily obtained from those in Exton *et al* (1969) on making the null rotation $\iota^A \to \iota^A - \bar{\omega}o^A$.

On writing $\lambda_1(r,\theta,\phi)=\overset{0}{\lambda}_1(\theta,\phi)+\overset{1}{\lambda}_1(\theta,\phi)r^{-1}+\overset{2}{\lambda}_1(\theta,\phi)r^{-2}+O(r^{-3})$, substituting into (45) and equating powers of r we obtain

$$-\bar{\eth}\overset{0}{\lambda}_0=\overset{0}{\lambda}_1 \tag{46}$$

$$\sigma^0\eth_0\overset{0}{\lambda}_0+\overset{0}{\lambda}_0\eth_0\bar{\sigma}^0=\overset{1}{\lambda}_1 \tag{47}$$

$$-\sigma^0\bar{\sigma}^0\bar{\eth}_0\overset{0}{\lambda}_0+(\tfrac{1}{2}\bar{\psi}^0_1-\bar{\sigma}^0\bar{\eth}_0\sigma^0)\overset{0}{\lambda}_0=\overset{2}{\lambda}_1+\sigma^0\bar{\sigma}^0\overset{0}{\lambda}_1. \tag{48}$$

For asymptotically constant spinors we have $\bar{\eth}_0\overset{0}{\lambda}_0+\overset{0}{\lambda}_1=0$ and $\eth_0\overset{0}{\lambda}_0=0$, so that (46) is automatically satisfied, (47) gives $\lambda_1=\overset{0}{\lambda}_0\eth_0\bar{\sigma}$ and (48) gives $\overset{2}{\lambda}_1=(\tfrac{1}{2}\bar{\psi}^0_1-\bar{\sigma}^0\bar{\eth}_0\sigma^0)\overset{0}{\lambda}_0$. We thus obtain the asymptotic expansions

$$\lambda_0(r,\theta,\phi)=\overset{0}{\lambda}_0(\theta,\phi) \tag{49}$$

$$\lambda_1(r,\theta,\phi)=-\bar{\eth}\overset{0}{\lambda}_0+\overset{0}{\lambda}_0\eth_0\bar{\sigma}^0r^{-1}+(\tfrac{1}{2}\bar{\psi}^0-\bar{\sigma}\bar{\eth}_0\sigma^0)\overset{0}{\lambda}_0r^{-2}+O(r^{-3}). \tag{50}$$

Substituting (43), (49), (50) into the RHS of (42) thus gives

$$\begin{aligned}\oint_{S_r}\lambda_0\lambda_1\,dS&=\oint[-\overset{0}{\lambda}_0\bar{\eth}\overset{0}{\lambda}_0r^2+\overset{0}{\lambda}_0\eth_0\bar{\sigma}^0r+(\tfrac{1}{2}\bar{\psi}^0_1-\bar{\sigma}^0\bar{\eth}\sigma^0)\overset{0}{\lambda}{}^2_0+\sigma^0\bar{\sigma}^0\overset{0}{\lambda}_0\bar{\eth}\overset{0}{\lambda}_0]\sin\theta\,d\theta\,d\phi\\&\quad+O(r^{-1})\\&=\oint\{-\tfrac{1}{2}r^2\bar{\eth}(\overset{0}{\lambda}{}^2_0)+r\eth_0(\overset{2}{\lambda}{}_0\bar{\sigma}^0)+\tfrac{1}{2}[\bar{\psi}_1-2\bar{\sigma}^0\bar{\eth}\sigma^0-\bar{\eth}(\sigma^0\bar{\sigma}^0)]\overset{0}{\lambda}{}^2_0\}\sin\theta\,d\theta\,d\phi\\&\quad+O(r^{-1}).\end{aligned} \tag{51}$$

The first two terms in the integrand vanish by the properties of the edth operator so that

$$\lim_{r\to\infty}\oint_{S_r}\lambda_0\lambda_1\,dS=\omega_{AB}\overset{0}{\lambda}{}^A\overset{0}{\lambda}{}^B$$

where

$$\omega_{AB}:=\oint_{S_\infty}(\bar{\psi}^0_1-2\bar{\sigma}^0\bar{\eth}\sigma^0-\bar{\eth}(\sigma^0\bar{\sigma}^0))o_Ao_B\,dS_0$$

which is just Bramson's (1975) expression for the spinor components of the angular momentum.

For a general cut S_r of $\mathcal{N}$ we now define the quasi-local angular momentum $\omega_{AB}(S_r)$ to be given by

$$\omega_{AB}(S_r)\overset{0}{\lambda}{}^A\overset{0}{\lambda}{}^B=\oint_{S_r}\lambda_{(A}\lambda_{B)}o^A\iota^B\,dS \tag{52}$$

where λ_A satisfies $\bar{\eth}\lambda_0+\rho\lambda_1=0$, $\eth_0\lambda_0=0$.

As we have seen, in the limit that S_r becomes a cut S_∞ of $\mathscr{I}^+$, $\omega_{AB}(S_r)$ agrees with Bramson's expression. It may also be shown that in the case of linearised gravity

$\omega_{AB}(S_r)$ reduces to the 'correct' value for the spinor components of the total angular momentum contained within S.

It is important to note that (42) is only well defined because of two important properties of the propagation equation that we have used. Firstly it determines $\overset{1}{\lambda}_1$ in such a way that the potentially unbounded $O(r)$ term in (51) in fact vanishes, and secondly because the $O(r^{-2})$ terms in the asymptotic expansion of λ_A are completely determined by the asymptotic components $\overset{0}{\lambda}_A$. This is not true, for example, of the 'Witten equation' on null surfaces that we used in our previous paper (Ludvigsen and Vickers 1982), since for that equation $\overset{2}{\lambda}_1(\theta, \phi)$ may be freely specified.

6. Discussion

Even though the methods we have used here cannot be used to give a general proof of the positivity of the Bondi mass, they provide a simple and direct proof in the special, but physically interesting, situations considered in § 4. The main reason for the simplicity of our method is the choice of null rather than space-like hypersurfaces. By restricting our attention to such surfaces, we have been able to replace the 'Witten equation' by a much simpler first-order propagation equation which also gives positivity. It is not altogether surprising that such a simplification occurs on null hypersurfaces as spinor methods are much more natural on such surfaces.

In § 3 we were able to show how to give a quasi-local definition of the energy–momentum associated with finite cuts of $\mathcal{N}$ in terms of a four-vector defined on $M_0 = \mathcal{S} \otimes \bar{\mathcal{S}}$. Furthermore, we showed that such a definition of quasi-local momentum possesses a number of desirable properties. In the situation where $\mathcal{N}$ converges to a single point, O, say, it is possible to give a different definition of local energy–momentum. Rather than specifying λ_A in terms of its asymptotic component, it is possible to determine λ_A over the whole of $\mathcal{N}$ by specifying λ_A at O. We can then define

$$\tilde{P}_{AA'}(S)\tilde{\lambda}^A\tilde{\lambda}^{A'} = \oint \phi_{AB} o^A \iota^B \, dS + \text{cc}$$

where $\tilde{\lambda}^A \in \mathcal{S}_0$ is the value of λ^A at O, $\tilde{P}_{AA'}(S) \in \mathcal{S}_0 \otimes \bar{\mathcal{S}}_0$ and $\mathcal{S}_0$ is the spin space at O. The drawback with such a definition is that it defines the quasi-local momentum as a four-vector in the Minkowski space $\tilde{M}_0 := \mathcal{S} \otimes \bar{\mathcal{S}}_0$ and there is, in general, no easy way to relate a vector in $\tilde{M}_0$ to one in $M_0 = \mathcal{S} \otimes \bar{\mathcal{S}}$, the Minkowski space of BMS translations. Such a definition of quasi-local momentum does however satisfy a mass-gain condition and reduces to the 'correct' expression in linearised gravity.

In § 5 we were able to show how to give a simple quasi-local definition of the angular momentum associated with finite cuts of $\mathcal{N}$. This definition has the property that in the limit that S becomes a cut of $\mathscr{I}^+$ it agrees with Bramson's expression. Unlike the case of the mass, this limit depends crucially upon the particular choice of propagation equation. That equation (15) may be used to give both a definition of quasi-local mass which satisfies a radial 'mass-gain' condition and a definition of quasi-local angular momentum with the right asymptotic behaviour is thus quite remarkable.

Finally, it is worth remarking upon certain similarities between the expressions in this paper and those given recently by Penrose (1982) for quasi-local mass and angular momentum which comes from a rather different approach to the problem based upon twistor theory.

Appendix

(*a*) *Proof that $P_a(S_\infty)$ gives the Bondi momentum.*
Let

$$I(r)=\oint_{S_r} \varphi_{AB}o^A\iota^B \,\mathrm{d}S+\mathrm{cc} \tag{53}$$

where φ_{AB}, o^A and ι^B are as in § 2. In terms of spin coefficients this may be written

$$I(r)=-\tfrac{1}{2}\oint_{S_r} [\lambda_{0'}\eth\lambda_1-\lambda_{1'}\bar{\eth}\lambda_0-\lambda_1\eth\lambda_{0'}+\lambda_0\bar{\eth}\lambda_{1'}-\lambda_0\lambda_{0'}(\mu+\bar{\mu})-\lambda_1\lambda_{1'}(\rho+\bar{\rho})]\,\mathrm{d}S. \tag{54}$$

Then by equation (6)

$$P_{\mathbf{AA'}}(S_\infty)\overset{0}{\lambda}{}^{\mathbf{A}}\overset{0}{\lambda}{}^{\mathbf{A'}}=\lim_{r\to\infty} I(r). \tag{55}$$

The simplest way of seeing that this gives the standard expression for the Bondi momentum is to substitute the asymptotic expansions of the relevant variables, in terms of Bondi coordinates, into equation (44). Remembering that our spinor dyad (o_A, ι_A) is related to a Bondi dyad (o^*_A, ι^*_B) by

$$o_A=o^*_A \qquad \iota_A=\iota^*_A-\bar{\omega}o_B \tag{56}$$

we obtain the following asymptotic expansions (Exton *et al* 1969):

$$\begin{aligned} &\rho=-r^{-1}+\mathrm{O}(r^{-3}) && \mu=-r^{-1}-(\sigma^0\dot{\bar{\sigma}}^0+\overset{0}{\psi}_2+\eth_0^2\bar{\sigma}^0)r^{-2}+\mathrm{O}(r^{-3})\\ &\eth=-r^{-1}\eth_0+\sigma^0r^{-2}\bar{\eth}_0+\mathrm{O}(r^{-3}) && \mathrm{d}S=\tfrac{1}{2}r^2(1-\sigma^0\bar{\sigma}^0r^{-2})\,\mathrm{d}S_0+\mathrm{O}(r^{-1})\end{aligned} \tag{57}$$

where $\eth_0$ is the Newman–Penrose edth operator on a metric sphere and $\mathrm{d}S_0=\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$.

Since λ_A is an asymptotically constant spinor

$$\lambda_0=\overset{0}{\lambda}_0+\overset{1}{\lambda}_0r^{-1}+\mathrm{O}(r^{-2}) \qquad \lambda_1=\overset{0}{\lambda}_1+\overset{1}{\lambda}_1r^{-1}+\mathrm{O}(r^{-2}) \tag{58}$$

where

$$\bar{\eth}_0\overset{0}{\lambda}_0=-\overset{0}{\lambda}_1 \qquad \text{and} \qquad \eth\overset{0}{\lambda}_0=0. \tag{59}$$

On substituting all the above in (44) and performing some integration by parts we obtain

$$I(r)=-\tfrac{1}{4}\oint(\sigma^0\dot{\bar{\sigma}}^0+\overset{0}{\psi}_2)\overset{0}{\lambda}_0\overset{0}{\lambda}_{0'}\,\mathrm{d}S_0+\mathrm{cc}+\mathrm{O}(r^{-1}) \tag{60}$$

which gives

$$P_{\mathbf{AA'}}(S_\infty)\overset{0}{\lambda}{}^{\mathbf{A}}\overset{0}{\lambda}{}^{\mathbf{A'}}=-\tfrac{1}{2}\oint(\sigma^0\dot{\bar{\sigma}}^0+\psi^0{}_2)\overset{0}{\lambda}_0\overset{0}{\lambda}_{0'}\,\mathrm{d}S_0 \tag{61}$$

i.e.

$$P_{\mathbf{AA'}}(S_\infty)=-\tfrac{1}{2}\oint(\sigma^0\dot{\bar{\sigma}}^0+\overset{0}{\psi}_2)o_{\mathbf{A}}o_{\mathbf{A'}}\,\mathrm{d}S_0. \tag{62}$$

(*b*) *Gauss's theorem for null hypersurfaces.*

Let $\mathcal{N}$ be a null hypersurface with affine parameter r and let $F_{ab} = \varepsilon_{AB}\bar{\varphi}_{A'B'} + \varepsilon_{A'B'}\varphi_{AB}$ be some real bivector. Let o^A and ι^A be defined as in § 2. Then the Gauss integral

$$I(r) = \oint_{S_r} F_{ab}\, d\Sigma^{ab} \tag{63}$$

may be rewritten as

$$I(r) = \oint_{S_r} \varphi_1\, dS + \text{cc} \tag{64}$$

where

$$\varphi_1 = \varphi_{AB}o^A\iota^B.$$

We now introduce coordinates (x^0, x^1, x^2, x^3) which are chosen so that $\mathcal{N}$ is given by $x^0 = 0$, $x^1 = r$ and x^2, x^3 label the null geodesics. Since $m^a = o^A\iota^{A'}$ is tangent to S_r, m^a has components $(0, 0, \xi^2, \xi^3)$. The two-surface metric is given by $g^s{}_{ab} = -2m_{(a}\bar{m}_{b)}$. It thus follows that

$$\det g_s{}^{ab} = -(\xi^2\bar{\xi}^3 - \xi^3\bar{\xi}^2)^2. \tag{65}$$

On using the relation

$$Þ\xi^i = \rho\xi^i + \sigma\xi^i \qquad i = 2, 3 \tag{66}$$

obtained from

$$(Þ\eth - \eth Þ)x^i = \rho\,\eth x^i + \sigma\bar{\eth} x^i \qquad i = 2, 3 \tag{67}$$

it may be shown that

$$Þ\, dS = -2\rho\, dS \tag{68}$$

and hence

$$dI/dr = \oint_{S_r} (Þ\varphi_1 - 2\rho\varphi_1)\, dS + \text{cc}. \tag{69}$$

On the other hand

$$l^a\nabla^b F_{ab} = (Þ\varphi_1 - 2\rho\varphi_1 - \bar{\eth}\varphi_0 + \tau'\varphi_0) + \text{cc} \tag{70}$$

where $\phi_0 = o^A o^B\varphi_{AB}$. But

$$\oint (\bar{\eth}\varphi_0 - \tau'\varphi_0)\, dS = \oint [m^a\nabla_a\varphi_0 - (m^a m^b\nabla_a m_b)\varphi_0]\, dS$$

$$= (i/\sqrt{2}) \oint \bar{\eth}_{NP}\varphi_0\, dS$$

$$= 0 \qquad \text{(since } \varphi_0 \text{ has spin weight} + 1)$$

where $\bar{\eth}_{NP}$ is the Newman–Penrose edth operator in the form given by Goldberg *et al* (1967).

Thus

$$dI/dr = \oint_{S_r} l^a\nabla^b F_{ab}\, dS$$

and hence

$$I(r_2)-I(r_1)=\int_{r=r_1}^{r_2}\oint_{S_r} l^a\nabla^b F_{ab}\,\mathrm{d}S\,\mathrm{d}r. \tag{71}$$

References

Bondi H, van der Burg M G J and Metzner A W K 1962 *Proc. R. Soc.* A **269** 21
Bramson B D 1975 *Proc. R. Soc.* A **341** 463
—— 1976 *Asymptotic Structure of Space–Time* ed P Esposito and L Witten (New York: Plenum)
Exton A R, Newman E T and Penrose R 1969 *J. Math. Phys.* **10** 1566
Geroch R, Held A and Penrose R 1973 *J. Math. Phys.* **14** 874
Goldberg J N, Macfarlane A J, Rohrlich F, Sudarshan E C G and Newman E T 1967 *J. Math. Phys.* **8** 2155
Horowitz G T and Perry M J 1982 *Phys. Rev. Lett.* **48** 371
Ludvigsen M and Vickers J A G 1982 *J. Phys. A: Math. Gen.* **15** L67
Penrose R 1968 *Battelle Rencontres* ed C de Witt and J Wheeler (New York: Benjamin)
—— 1982 *Proc. R. Soc.* A **381** 53
Schoen R and Yau S T 1982 *Phys. Rev. Lett.* **48** 371
Witten E 1981 *Commun. Math. Phys.* **80** 381